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LETTER TO THE EDITOR

Quantum gaps and classical orbits in a rotating two-dimensional harmonic oscillator

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Abstract. The spectrum of a particle in a rotating two-dimensional harmonic oscillator is studied as a function of the cranking frequency. The occurrence of energy gaps is linked to classical periodic orbits. Special attention is paid to the Farey fan pattern generated beyond the Landau limit.

In this letter, we present the patterns of the quantum gaps that are formed in the energy spectrum of a rotating two-dimensional harmonic oscillator as the cranking frequency is varied. Although this mathematical model is very elementary and is fully solvable both quantum mechanically and classically, there are intriguing points that emerge from the study, particularly regarding the lifting of the degeneracies beyond the Landau level limit. Furthermore, this is perhaps the simplest quantum model that generates the Farey fan pattern [1], showing clearly the intimate connection between quantum gaps and classical periodic orbits [2].

Consider the two-dimensional motion of a particle on the xy-plane, governed by the Hamiltonian

$$H = \frac{1}{2M}(p_x^2 + p_y^2) + \tfrac{1}{2}M\Omega^2(x^2+y^2) + \omega(xp_y - yp_x)\,. \tag{1}$$

The particle is in a harmonic oscillator of frequency Ω, and the oscillator itself is rotating about the negative z-axis with a frequency ω. Since the last term in (1), ωl_z, commutes with the harmonic oscillator Hamiltonian, the eigenspectrum of H may be immediately written down using polar coordinates

$$E_{n_r,l} = (2n_r + |l| + 1)\hbar\Omega + l\hbar\omega\,. \tag{2}$$

Here n_r is the radial quantum number, with values $n_r = 0, 1, 2, \ldots$, and $\hbar l$ is the angular momentum along the z-axis, with $l = 0, \pm 1, \pm 2$, etc. In figure 1, the pattern of the energy levels generated by (2) is shown by varying the cranking frequency ω in the range $0 \leqslant \omega \leqslant 2\Omega$, keeping the oscillator frequency Ω fixed. We define

$$\tilde{\omega} = (\omega - \Omega) \qquad \nu = \frac{\tilde{\omega}}{2\Omega} \tag{3}$$

and plot the energy levels as a function of the dimensionless quantity ν in the range $-\frac{1}{2} \leqslant \nu \leqslant \frac{1}{2}$ in figure 1. The collapse of the single-particle states for $\omega = \Omega$ into highly degenerate equally spaced levels is clearly seen, and these are called Landau

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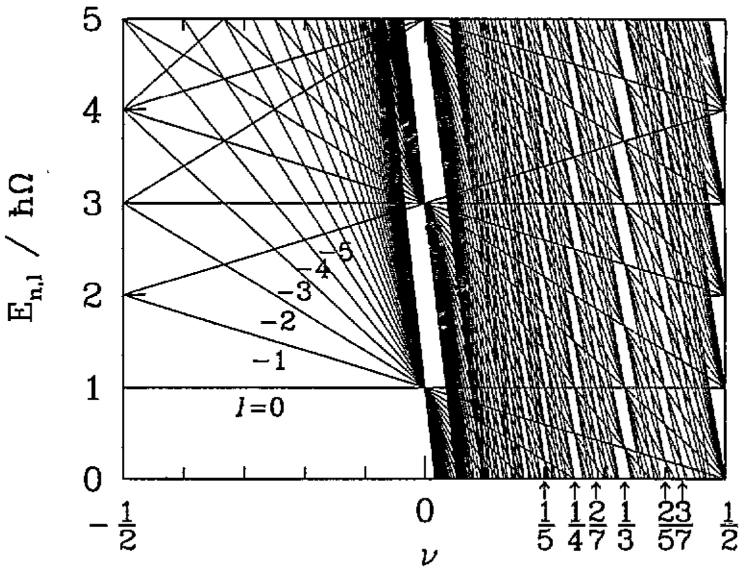

Figure 1. The energy spectrum (2) of a cranked two-dimensional harmonic oscillator, shown as a function of ν, as defined in (3). The Landau levels are formed at $\nu = 0$.

levels [3] in the context of the motion of a charged particle in a uniform magnetic field. For such a particle (ignoring intrinsic spin), the Hamiltonian is $H_B = (1/2M)(\boldsymbol{p} - (e/c)\boldsymbol{A})^2$, where e is the charge, and $\boldsymbol{p}$ the momentum for motion in the xy-plane. The vector potential $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{B} \times \boldsymbol{r})$ generates a transverse magnetic field $\boldsymbol{B}$. Taking $e = -|e|$, and the symmetric gauge, $\boldsymbol{A} = (-\frac{1}{2}By, \frac{1}{2}Bx, 0)$, H_B reduces to H of (1) for $\Omega = \omega = \omega_c/2$, where $\omega_c = |e|B/Mc$ is called the cyclotron frequency. As is clear from figure 1, the collapsing single-particle states in the lowest Landau level all have aligned (but different) angular momenta, and originate from different shells of the harmonic oscillator. Although this result is well known in the literature, its diagrammatic depiction through a cranked oscillator is not commonly shown. The pattern in figure 1 for $\tilde{\omega} > 0$ is new, and its study will be the main topic of the letter.

Even for the range $-\frac{1}{2} \leqslant \nu < 0$, the convergence of the states to the Landau levels is preceded by a zig-zag repeating pattern of gaps. This is shown vividly for the higher excited states in figure 2. Note that the quantum gaps appear at those values of the cranking frequency for which ω/Ω is a rational number, a fact that links these gaps to the occurrence of classical periodic orbits. These may be easily seen by examining the solutions of the classical equations of motion. Using the variable $z = x + \mathrm{i}y$, the two equations of motion may be written compactly as

$$\ddot{z} = \left(\omega^2 - \Omega^2\right)z + 2\mathrm{i}\omega\dot{z}\,. \tag{4}$$

Note that the first term on the right is an attractive harmonic force for $\omega < \Omega$, but becomes repulsive for cranking frequencies beyond the Landau levels. The general solution of (4) is

$$z = A\mathrm{e}^{\mathrm{i}(\omega-\Omega)t} + B\mathrm{e}^{\mathrm{i}(\omega+\Omega)t} \tag{5}$$

where A and B are constants. The two normal mode frequencies (for $\omega \neq \Omega$) are $|\Omega - \omega|$ and $(\Omega + \omega)$. When the ratio of these frequencies is a rational fraction, a closed periodic orbit in the xy-plane is obtained. The accidental degeneracy of the corresponding quantum

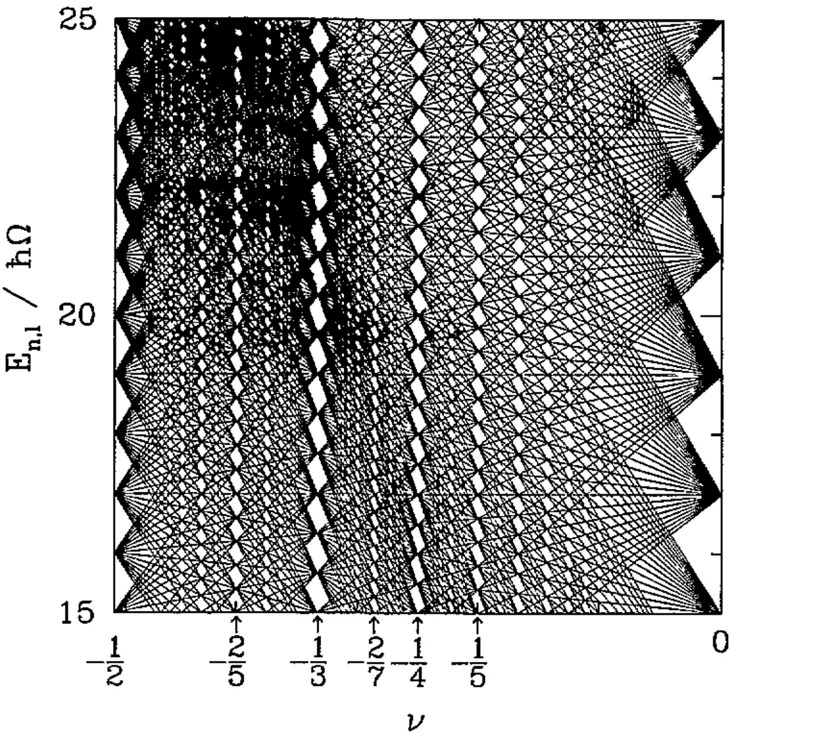

Figure 2. The same spectrum as figure 1, in the range $-\frac{1}{2} < \nu < 0$ for higher excitation energies.

problem (for $\Omega > \omega$) has been discussed by Louck *et al* [4]. Some of these periodic orbits are shown in figure 3 for various values of ν. For the special case of $\omega = \Omega$, the solutions of (4) are given by circles in the xy-plane with arbitrary centres. The quantum analogue of this motion with coherent states is analysed in [5, 6]. Apart from this case, the most prominent gaps occur in figures 1 and 2 for the simplest fractions. For example, the large gaps at $\nu = \pm\frac{1}{3}$ correspond to the situation when one normal mode frequency is twice the other.

It is interesting to study the degeneracy of the levels for the situation when the cranking frequency ω is greater than or equal to the oscillator frequency Ω. For convenience, let us study the states converging at the energy of the lowest Landau level, $E = \hbar\Omega$. At this energy, at $\omega = \Omega$, the nodeless ($n_r = 0$, see (2)) angular momentum states of the largest negative values from each shell converge. The resulting degeneracy per unit area is easily calculated by summing the squares of the normalized 'stretched' single-particle states. This is found to be $\eta_0 = 2M\Omega/h$, which equals eB/hc if Ω is chosen to be $\omega_c/2$, leading to a well known result. Now let us proceed to examine, in figure 1, the degeneracies at the gaps to the right of the Landau levels. The repeating pattern in this region is known as a Farey fan [1], and has been studied in the context of number theory and continued fractions. At the energy $E = \hbar\Omega$, inspection of the level at $\nu = 1/m$ (m an integer > 1) reveals that the number of converging single-particle states is exactly a fraction $1/m$ of the Landau level. For example, the successive harmonic oscillator states meeting at $\nu = \frac{1}{3}$ at $E = \hbar\Omega$ have angular momenta $l = 0, -3, -6$, etc. in units of $\hbar$ (see figure 4). Thus, for every triplet of adjacent states in the lowest Landau level, e.g. $(0, -1, -2)$, there is

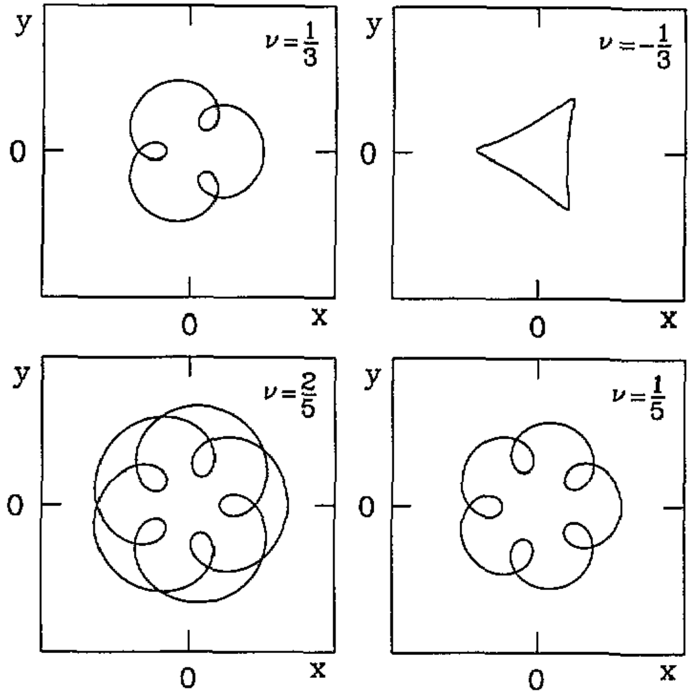

Figure 3. The classical periodic orbits of a particle obeying (5) for various rational ν.

one ($l = 0$ in this case) at $\nu = \frac{1}{3}$. Similarly, at $\tilde{\omega}/2\Omega = \frac{1}{5}$, the degeneracy/area is $\frac{1}{5}\eta_0$. The collapsed single-particle states at such $\nu = 1/m$ gaps are each from a separate Landau level, and have increasing number of nodes. For example, at $\nu = \frac{1}{5}$, the $l = 0$ state has no node, the next state with $l = -5$ has one node, the $l = -10$ state has two nodes, and so on. We may term the condensed levels at $\nu = 1/m$ as 'mothers', since these gaps rise to a succession of 'daughters' as seen in figure 4. One state from each quantum shell of the $\nu = \frac{1}{3}$ mother converges at $E = \hbar\Omega$, constituting the daughter at $\frac{2}{5}$, just as the mother herself was formed from the collapse of the states from each Landau level. The Landau level, in turn, was formed by the collapse of the states from the separate oscillator shells. The single-particle states converging at the daughters have different nodal structures than the mothers. Consider, for example, the daughters at $\nu = \frac{2}{5}$ and $\frac{3}{7}$ at $E = \hbar\Omega$ that belong to the mother at $\nu = \frac{1}{3}$. Using equation (2), all the converging states at this energy obey the equation

$$(2n_r + |l|)\hbar\Omega + l\hbar\omega = 0. \tag{6}$$

It immediately follows that the converging states at $\nu = \frac{2}{5}$ have $n_r = 2$ for $l = -5$, $n_r = 4$ for $l = -10$, etc. Similarly, for $\nu = \frac{3}{7}$, $n_r = 3$ for $l = -7$, $n_r = 6$ for $l = -14$, etc. A similar construction could be made with even-denominator mothers, but then the daughters have alternately even and odd denominators. The complexity of the structure in the quantum states is reflected in the classical periodic orbits also, some of which are shown in figure 3. From this figure, note that the number of loops in the orbit is determined by the denominator q in $\nu = p/q$. For example, both $\nu = \frac{1}{5}$ and $\nu = \frac{2}{5}$ have periodic orbits with five loops, but the $\nu = \frac{2}{5}$ orbit has a more complicated structure. The denominator q of $\nu = p/q$ also determines the magnitude

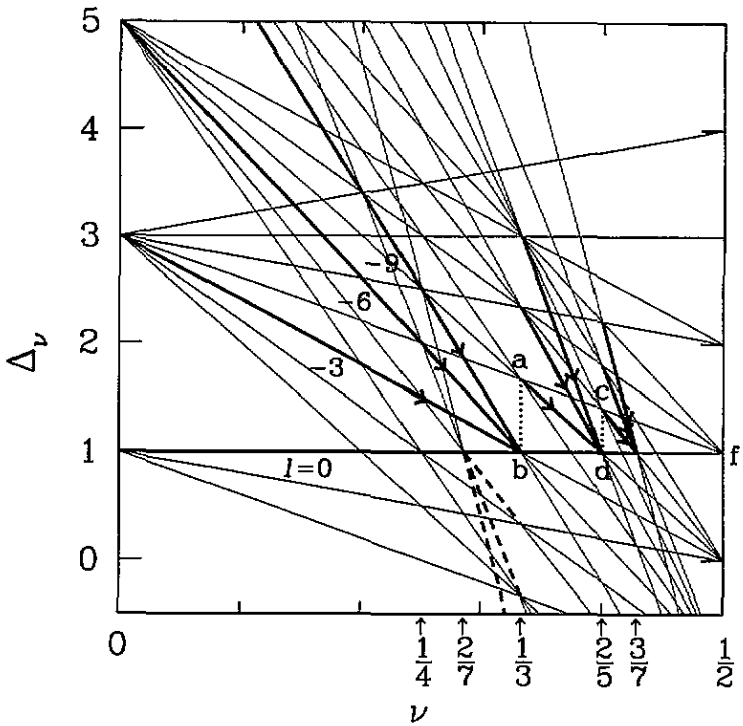

Figure 4. The 'mother–daughter' sequence of degenerate levels. One state from each Landau level converges at $\nu = \frac{1}{3}$ to form a mother. Similarly, one state from each shell at $\frac{1}{3}$ converges at $\frac{2}{5}$. Only a few converging lines are shown for clarity. In the triangle abf, the vertical lines ab and cd show the gaps at $\nu = \frac{1}{3}$ and $\frac{2}{5}$, respectively.

of the quantum gap, as is apparent from figure 4. Denoting this gap by Δ, we see that at $\nu = p/q$,

$$\Delta_\nu = \frac{\hbar\omega_c}{q} \tag{7}$$

where $\hbar\omega_c = 2\hbar\Omega$ is the gap at the Landau levels.

Finally it may be worth mentioning that the sequence of the gaps generated by this dynamical model for $\nu > 0$ are the same, for the odd denominators, as the Haldane hierarchy [7] in the fractional quantum Hall effect [8]. The hole-state sequences for the odd denominators in this hierarchy, e.g. $(\frac{2}{7}, \frac{3}{11}, \frac{4}{15}, \ldots)$ are generated in our model by the convergence of lines from the lower side of the $\Delta_\nu = 1$ line (see figure 4). The FQHE states, however, have a very different structure than the ones that have been generated by our simple model. In FQHE, the single-particle states of the lowest Landau level get thoroughly mixed by the Coulomb interaction between the electrons, and have a highly correlated wavefunction of an incompressible quantum fluid [9]. There is little mixing of states from different Landau levels in such a state. By contrast, the wavefunction generated by our model has thorough inter-Landau level mixing, and has no two-body correlations. Nevertheless, it is interesting that a sequence of quantum gaps resembling the Haldane ones may be produced from an independent particle model that is integrable.

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